

Free filtrations of affine Weyl arrangements and the ideal-Shi arrangements

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Abstract

In this article we prove that the ideal-Shi arrangements are free central arrangements of hyperplanes satisfying the dual-partition formula. Then it immediately follows that there exists a saturated free filtration of the cone of any affine Weyl arrangement such that each filter is a free subarrangement satisfying the dual-partition formula. This generalizes the main result in [1] which affirmatively settled a conjecture by Sommers and Tymoczko [9].

1 Introduction

Let Φ be an irreducible crystallographic root system of rank ℓ over the real number field \mathbb{R} . Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be a simple system of Φ and Φ^+ the corresponding positive system. An **ideal** $I \subseteq \Phi^+$ is a set such that if $\alpha \in I$, $\beta \in \Phi^+$ with $\alpha - \beta \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0} \alpha_i$, then $\beta \in I$. For any subset Σ of Φ^+ , define $\mathcal{A}(\Sigma) := \{H_\alpha \mid \alpha \in \Sigma\}$, where H_α is the hyperplane perpendicular to α in the ℓ -dimensional Euclidean space.

Let V be the $(\ell + 1)$ -dimensional Euclidean space with a basis $\Delta \cup \{z\}$, where z is a unit vector perpendicular to each α_i ($1 \leq i \leq \ell$). We usually identify V with its dual space V^* by the inner product. For $j \in \mathbb{Z}$ and $\alpha \in \Phi^+$, define a hyperplane $H_\alpha^j := \{\alpha - jz = 0\}$ in V . Let H_z denote the hyperplane in V defined by $\{z = 0\}$.

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Definition 1.1

For $k \in \mathbb{Z}_{>0}$ and an ideal $I \subseteq \Phi^+$, define the **ideal-Shi** arrangements in V by

$$\begin{aligned}\mathcal{S}_{+I}^k &:= \{H_\alpha^j \mid \alpha \in \Phi^+, -k+1 \leq j \leq k\} \cup \{H_z\} \cup \{H_\alpha^{-k} \mid \alpha \in I\}, \\ \mathcal{S}_{-I}^k &:= (\{H_\alpha^j \mid \alpha \in \Phi^+, -k+1 \leq j \leq k\} \cup \{H_z\}) \setminus \{H_\alpha^k \mid \alpha \in I\}.\end{aligned}$$

The following Theorems 1.2, 1.3 and 1.5 are the main theorems of this article. They concern the **freeness** and the **exponents** of the ideal-Shi arrangements. (See § 2 for the terminology of the theory of (free) arrangements of hyperplanes.)

Theorem 1.2

All the ideal-Shi arrangements $\mathcal{S}_{\pm I}^k$ are free.

To determine the exponents of ideal-Shi arrangements, we define the **dual partition**: Let Σ be a finite set of vectors in a d -dimensional vector space and $f : \Sigma \rightarrow \mathbb{Z}_{>0}$ be a function such that $f_i := |f^{-1}(i)| \leq f_{i-1} = |f^{-1}(i-1)| \leq d$ for all $i \in \mathbb{Z}_{>1}$. Define $m := \max_{s \in \Sigma} \{f(s)\}$. Then the dual partition of the pair (Σ, f) is the set of integers

$$((0)^{d-f_1}, (1)^{f_1-f_2}, (2)^{f_2-f_3}, \dots, (m-1)^{f_{m-1}-f_m}, (m)^{f_m}),$$

where $(a)^b$ ($a, b \in \mathbb{Z}_{\geq 0}$) indicates that there are b copies of a . The most famous example of the dual partition is the case when $\Sigma = \Phi^+$ and $f = \text{ht} : \Phi^+ \rightarrow \mathbb{Z}_{>0}$, where ht is the height function defined by $\text{ht}(\sum_{i=1}^\ell c_i \alpha_i) = \sum_{i=1}^\ell c_i$. In this case, the dual partition of the pair (Φ^+, ht) is equal to the exponents of the root system [10] [6] [7]. This remarkable dual-partition formula is generalized to the ideal-Shi arrangements as follows:

Theorem 1.3

Let $k \in \mathbb{Z}_{>0}$ and $I \subseteq \Phi^+$ be an ideal. Denote the Coxeter number of Φ by h . For $\alpha \in \Phi^+$ and $j \in \mathbb{Z}$, define the extended height function $\tilde{\text{ht}}$ by

$$\tilde{\text{ht}}(\alpha - jz) = \begin{cases} -\text{ht}(\alpha) + jh + 1 & \text{if } j > 0, \\ \text{ht}(\alpha) - jh & \text{if } j \leq 0. \end{cases}$$

We also define $\tilde{\text{ht}}(z) = 1$. Then

(1) the exponents of \mathcal{S}_{+I}^k is the dual partition of the pair

$$(\{\alpha - jz \mid \alpha \in \Phi^+, -k+1 \leq j \leq k\} \cup \{\alpha + kz \mid \alpha \in I\} \cup \{z\}, \tilde{\text{ht}}),$$

(2) the exponents of \mathcal{S}_{-I}^k is the dual partition of the pair

$$(\{\alpha - jz \mid \alpha \in \Phi^+, -k+1 \leq j \leq k\} \setminus \{\alpha - kz \mid \alpha \in I\} \cup \{z\}, \tilde{\text{ht}}).$$

Remark 1.4

Note that $\mathcal{S}_{\pm I}^k$ are equal to the cones (i.e., [8, Definition 1.15]) of the (extended and generalized) Shi arrangement \mathcal{S}^k when I is the empty set. Also note that \mathcal{S}_{+I}^k is equal to the cones of the (extended and generalized) Catalan arrangement Cat^k when $I = \Phi^+$. In these cases, Theorems 1.2 and 1.3 had been conjectured by Edelman and Reiner in [5] before they were proved by Yoshinaga in [11].

For a central arrangement \mathcal{A} of countably infinite hyperplanes, we say that a filtration

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_i \subseteq \cdots \quad \text{with} \quad \mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i$$

of \mathcal{A} is said to be **saturated** if $|\mathcal{A}_i| = i$ for any $i \in \mathbb{Z}_{>0}$. We also say that the filtration is **free** if each \mathcal{A}_i is a free arrangement. Then Theorems 1.2 and 1.3 immediately imply the following:

Theorem 1.5

For a root system Φ , fix a linear order $(\alpha_1, \dots, \alpha_n)$ on the set Φ^+ of positive roots in such a way that $\{\alpha_i\}_{i=1}^k$ is an ideal of Φ^+ for any $1 \leq k \leq n$. Define

$$H_i^j := H_{\alpha_i}^j = \{\alpha_i - jz = 0\} \quad (j \in \mathbb{Z}, 1 \leq i \leq n)$$

and

$$K_p := \begin{cases} H_r^{-q} & \text{if } 1 \leq r \leq n, \\ H_{2n+1-r}^{q+1} & \text{if } n+1 \leq r \leq 2n, \end{cases}$$

where $p \in \mathbb{Z}_{>0}$ with $p = 2nq + r$ ($1 \leq r \leq 2n$, $q \in \mathbb{Z}_{\geq 0}$). Let

$$\mathcal{A}_i := \{H_z, K_1, K_2, \dots, K_{i-1}\} \quad (i \in \mathbb{Z}_{>0}).$$

Then the filtration $\mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_i \subseteq \cdots$ of the cone

$$\mathcal{A}_{\infty}(\Phi) := \{H_z\} \cup \{H_i^j \mid j \in \mathbb{Z}, 1 \leq i \leq n\}$$

of the affine Weyl arrangement is saturated and free. Moreover, the exponents of \mathcal{A}_i is the dual partition of the pair

$$(\{z\} \cup \{\alpha - jz \mid \alpha \in \Phi^+, j \in \mathbb{Z}, \{\alpha - jz = 0\} \in \mathcal{A}_i\}, \tilde{\text{ht}}).$$

In Theorems 1.2 and 1.3, we considered two types of ideal-Shi arrangements \mathcal{S}_{+I}^k and \mathcal{S}_{-I}^k . In fact, for an arbitrary subset Σ of Φ^+ , we have the following theorem asserting a symmetry of the freeness and the exponents with respect to \mathcal{S}^k .

Theorem 1.6

For $k \in \mathbb{Z}_{>0}$ and an arbitrary subset $\Sigma \subset \Phi^+$, define

$$\begin{aligned}\mathcal{S}_{+\Sigma}^k &:= \{H_\alpha^j \mid \alpha \in \Phi^+, -k+1 \leq j \leq k\} \cup \{H_z\} \cup \{H_\alpha^{-k} \mid \alpha \in \Sigma\}, \\ \mathcal{S}_{-\Sigma}^k &:= (\{H_\alpha^j \mid \alpha \in \Phi^+, -k+1 \leq j \leq k\} \cup \{H_z\}) \setminus \{H_\alpha^k \mid \alpha \in \Sigma\}.\end{aligned}$$

Then the arrangement $\mathcal{S}_{+\Sigma}^k$ is free with exponents $(1, kh + m_1, \dots, kh + m_\ell)$ if and only if the arrangement $\mathcal{S}_{-\Sigma}^k$ is free with exponents $(1, kh - m_1, \dots, kh - m_\ell)$. In this case, the arrangement $\mathcal{A}(\Sigma)$ is also free with exponents (m_1, \dots, m_ℓ) .

The organization of this article is as follows. In §2 we present the four earlier results which will play important roles in the subsequent sections. They are the two freeness criteria (Theorem 2.1 in [12] and Theorem 2.2 in [4]), the ideal free theorem (Theorem 2.4, [1]), and the shift isomorphism (Theorem 2.3 in [3]). In §3 we prove Theorem 1.6 after we study root systems of rank two. In §4 we prove Theorems 1.2 and 1.3.

2 Preliminaries

In this section let \mathcal{A} be a central arrangement of hyperplanes in $V = \mathbb{R}^n$, i.e., a finite set of hyperplanes of V going through the origin. For each $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such that $\ker \alpha_H = H$. Let $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$. An **intersection poset** $L(\mathcal{A})$ is defined by

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\}, \quad L_k(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \text{codim } X = k\} \quad (k \in \mathbb{Z}_{\geq 0}).$$

Then, ordered by reverse inclusion, $L(\mathcal{A})$ is a poset with the minimum element V . The **characteristic polynomial** $\chi(\mathcal{A})$ is defined by

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X},$$

where the **Möbius function** $\mu : L(\mathcal{A}) \rightarrow \mathbb{Z}$ is defined by

$$\mu(X) = \begin{cases} 1 & (X = V), \\ -\sum_{V \supset Y \supsetneq X} \mu(Y) & (X \neq V). \end{cases}$$

Since \mathcal{A} is central, it is known that $\chi(\mathcal{A}, t)$ is divisible by $t - 1$. Define $\chi_0(\mathcal{A}, t) := \chi(\mathcal{A}, t)/(t - 1)$.

For $X \in L(\mathcal{A})$, define

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}, \quad \mathcal{A}^X := \{K \cap X \mid K \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

Let $S = S(V^*)$ be a symmetric algebra of V^* , $\text{Der } S$ the derivation module of S and Ω_S^q the S -module of regular differential q -forms. Define

$$\begin{aligned} D(\mathcal{A}) &= \{\theta \in \text{Der } S \mid \theta(\alpha_H) \in S\alpha_H \text{ for all } H \in \mathcal{A}\}, \\ \Omega^q(\mathcal{A}) &= \{\omega \in (1/Q(\mathcal{A}))\Omega_S^1 \mid Q(\mathcal{A})\omega \wedge d\alpha_H \in \alpha_H\Omega_S^{q+1} \text{ for all } H \in \mathcal{A}\}. \end{aligned}$$

It is known (e. g., [8]) that the S -modules $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are dual to each other. We say that \mathcal{A} is **free with exponents** $\exp(\mathcal{A}) = (d_1, \dots, d_n)$ if there are homogeneous derivations $\theta_1, \dots, \theta_n \in D(\mathcal{A})$ such that $D(\mathcal{A}) = \oplus_{i=1}^n S\theta_i$ with $\deg \theta_i = d_i$ ($i = 1, \dots, n$). By the duality above, \mathcal{A} is free with exponents (d_1, \dots, d_n) if and only if $\Omega^1(\mathcal{A})$ is a free S -module of rank n with homogeneous basis $\omega_1, \dots, \omega_n$ such that $\deg \omega_i = -d_i$ ($i = 1, \dots, n$).

Finally, let us introduce four key results to prove Theorem 1.2. To state them, let us introduce multiarrangements. For a central arrangement \mathcal{A} and $m : \mathcal{A} \rightarrow \mathbb{Z}_{>0}$, the pair (\mathcal{A}, m) is called a **multiarrangement**. Define

$$D(\mathcal{A}, m) := \{\theta \in \text{Der } S \mid \theta(\alpha_H) \in S\alpha_H^{m(H)} \text{ for all } H \in \mathcal{A}\}.$$

Also, the freeness of (\mathcal{A}, m) and the exponents $\exp(\mathcal{A}, m)$ can be defined in the same way as the freeness of \mathcal{A} and $\exp(\mathcal{A})$. For a fixed $H_0 \in \mathcal{A}$, define $m_0 : \mathcal{A}^{H_0} \rightarrow \mathbb{Z}_{>0}$ by

$$m_0(H \cap H_0) := |\{K \in \mathcal{A} \setminus \{H_0\} \mid K \cap H_0 = H \cap H_0\}|.$$

The multiarrangement (\mathcal{A}^{H_0}, m_0) is called the **Ziegler restriction** of \mathcal{A} onto H_0 . If \mathcal{A} is free with $\exp(\mathcal{A}) = (1, d_2, \dots, d_n)$, then (\mathcal{A}^{H_0}, m_0) is free with $\exp(\mathcal{A}^{H_0}, m_0) = (d_2, \dots, d_n)$. For $D_0(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(\alpha_{H_0}) = 0\}$, define the **Ziegler restriction map** $D_0(\mathcal{A}) \rightarrow D(\mathcal{A}^{H_0}, m_0)$ as the restriction of a derivation onto H_0 . For details, see [13].

Theorem 2.1 ([12], Theorem 3.2)

Let \mathcal{A} be a central arrangement in \mathbb{R}^3 , $H_0 \in \mathcal{A}$ and (\mathcal{A}'', m) the Ziegler restriction of \mathcal{A} onto H_0 . Let $\exp(\mathcal{A}'', m) = (d_1, d_2)$. Then \mathcal{A} is free with $\exp(\mathcal{A}) = (1, d_1, d_2)$ if and only if $\chi_0(\mathcal{A}, 0) = d_1 d_2$.

Theorem 2.2 ([4], Theorem 4.1)

Let \mathcal{A} be a central arrangement in \mathbb{R}^n ($n > 3$) and fix $H_0 \in \mathcal{A}$. Let (\mathcal{A}'', m) be the Ziegler restriction of \mathcal{A} onto H_0 . Assume that

- (1) (\mathcal{A}'', m) is free, and
- (2) \mathcal{A}_X is free for any $X \in L_3(\mathcal{A})$ with $X \subset H_0$.

Then \mathcal{A} is free.

We use the notation from §1: let Φ be an irreducible crystallographic root system of rank ℓ .

Theorem 2.3 (Shift isomorphism, [3], Corollary 12)

Let $k \in \mathbb{Z}_{>0}$, $\mathcal{A} := \mathcal{A}(\Phi^+)$ and $m : \mathcal{A} \rightarrow \{0, 1\}$ be a multiplicity. Then there exist isomorphisms of S -modules

$$D(\mathcal{A}, m) \rightarrow D(\mathcal{A}, 2k + m), \quad \Omega^1(\mathcal{A}, m) \rightarrow D(\mathcal{A}, 2k - m).$$

Hence if (\mathcal{A}, m) is free with $\exp(\mathcal{A}, m) = (m_1, \dots, m_\ell)$, then $(\mathcal{A}, 2k \pm m)$ is also free with $\exp(\mathcal{A}, 2k \pm m) = (kh \pm m_1, \dots, kh \pm m_\ell)$.

In [9] Sommers and Tymoczko posed the conjecture corresponding to Theorems 1.2 and 1.3 for $k = 0$. The conjecture was affirmatively settled as follows:

Theorem 2.4 (Ideal free theorem, [1], Theorem 1.1)

Let $I \subseteq \Phi^+$ be an ideal. Then $\mathcal{A}(I) = \{H_\alpha \mid \alpha \in I\}$ is a free arrangement and its exponents $(m_1(I), m_2(I), \dots, m_\ell(I))$ are equal to the dual partition of the pair (I, ht) , where ht is the height function of positive roots.

3 Proof of Theorem 1.6

In this section we continue to use the notation from §1. Before the proof of Theorem 1.6, we will verify Lemma 3.1 and then prove Proposition 3.2 which is a key to the proof of Theorem 1.6. In Lemma 3.1 and Proposition 3.2, let Φ denote an irreducible crystallographic root system of rank two (i.e., $\Phi = A_2, B_2$ or G_2). Recall that Δ is a simple system of Φ . Fix $k \in \mathbb{Z}_{>0}$.

Lemma 3.1

For $\alpha, \beta \in \Phi^+$ ($\alpha \neq \beta$), let $p_- := H_\alpha^k \cap H_\beta^k$.

(1) If $\Delta = \{\alpha, \beta\}$, then $\{H_\gamma^s \mid \gamma \in \Phi^+, -k \leq s \leq k, p_- \subset H_\gamma^s\} = \{H_\alpha^k, H_\beta^k\}$,

(2) if $\Delta \neq \{\alpha, \beta\}$, then there exists $\gamma \in \Phi^+$ such that $p_- \subset H_\gamma^0$.

These two results hold true also for $p_+ := H_\alpha^{-k} \cap H_\beta^{-k}$.

Proof. (1) Assume that $\Delta = \{\alpha, \beta\}$ and that $p_- \subset H_\gamma^s$ for some $\gamma \in \Phi^+$ and some s with $-k \leq s \leq k$. Then we have

$$a(\alpha - kz) + b(\beta - kz) = \gamma - sz$$

for some nonzero rational numbers a, b . Since $a\alpha + b\beta = \gamma$, one has $\{a, b\} \subset \mathbb{Z}_{>0}$. Thus $s = ak + bk = (a + b)k > k$, which is a contradiction.

(2) If $\Delta \neq \{\alpha, \beta\}$, by case-by-case arguments for A_2, B_2, G_2 , we have $\alpha - \beta \in \mathbb{Z}\gamma$ for some $\gamma \in \Phi^+$. This implies $p_- = H_\alpha^k \cap H_\beta^k \subset H_\gamma^0$ because $(\alpha - kz) - (\beta - kz) \in \mathbb{Z}\gamma$.

In the case of p_+ , the parallel proof works. \square

For an arbitrary arrangement \mathcal{A} and a hyperplane H_0 , define

$$\mathcal{A} \cap H_0 := \{K \cap H_0 \mid K \in \mathcal{A}, K \neq H_0\}.$$

Then $\mathcal{A} \cap H_0$ is an arrangement in H_0 .

Proposition 3.2

Let $\Sigma \subseteq \Phi^+$ and $\alpha \in \Phi^+ \setminus \Sigma$. Then

(1)

$$|\mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}| = \begin{cases} kh + 1 & \text{if } \alpha \in \Delta, \Sigma \cap \Delta = \emptyset, \\ kh + 2 & \text{otherwise,} \end{cases}$$

(2)

$$|\mathcal{S}_{-\Sigma}^k \cap H_\alpha^k| = \begin{cases} kh + 1 & \text{if } \alpha \in \Delta, \Sigma \cap \Delta = \emptyset, \\ kh & \text{otherwise.} \end{cases}$$

Proof. When $\Sigma = \emptyset$, by directly counting the intersections, we get the following equalities ([2]):

$$(3.1) \quad |\mathcal{S}^k \cap H_\alpha^{-k}| = \begin{cases} kh + 1 & \text{if } \alpha \in \Delta, \\ kh + 2 & \text{otherwise,} \end{cases}$$

$$|\mathcal{S}^k \cap H_\alpha^k| = \begin{cases} kh + 1 & \text{if } \alpha \in \Delta, \\ kh & \text{otherwise.} \end{cases}$$

(1) Consider the difference set

$$D_+ := (\mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}) \setminus (\mathcal{S}^k \cap H_\alpha^{-k}).$$

Case 1. Suppose $\alpha \notin \Delta$. Let $\beta \in \Sigma$. Then $p_+ = H_\beta^{-k} \cap H_\alpha^{-k} \subset H_\gamma^0$ for some $\gamma \in \Phi^+$ by Lemma 3.1 (2). This implies

$$p_+ = H_\beta^{-k} \cap H_\alpha^{-k} = H_\gamma^0 \cap H_\alpha^{-k} \in \mathcal{S}^k \cap H_\alpha^{-k}.$$

Therefore $D_+ = \emptyset$.

Case 2. Suppose $\alpha \in \Delta$ and $\Sigma \cap \Delta = \emptyset$. Then an arbitrary root $\beta \in \Sigma$ is non-simple. Thus $p_+ = H_\beta^{-k} \cap H_\alpha^{-k} \subset H_\gamma^0$ for some $\gamma \in \Phi^+$ by Lemma 3.1 (2). This implies

$$p_+ = H_\beta^{-k} \cap H_\alpha^{-k} = H_\gamma^0 \cap H_\alpha^{-k} \in \mathcal{S}^k \cap H_\alpha^{-k}.$$

Therefore $D_+ = \emptyset$.

Case 3. Suppose $\alpha \in \Delta$ and $\Sigma \cap \Delta \neq \emptyset$. Then we may express $\Delta = \{\alpha, \beta\}$ with $\beta \in \Sigma$. By Lemma 3.1 (1) we have

$$p_+ = H_\beta^{-k} \cap H_\alpha^{-k} \in \mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}, \quad p_+ = H_\beta^{-k} \cap H_\alpha^{-k} \notin \mathcal{S}^k \cap H_\alpha^{-k}.$$

Therefore $D_+ = \{p_+\}$.

Combining (3.1) with the three cases above, we get

$$|\mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}| = \begin{cases} |\mathcal{S}^k \cap H_\alpha^{-k}| = kh + 1 & \text{if } \alpha \in \Delta, \Sigma \cap \Delta = \emptyset, \\ |\mathcal{S}^k \cap H_\alpha^{-k}| + 1 = kh + 2 & \text{if } \alpha \in \Delta, \Sigma \cap \Delta \neq \emptyset, \\ |\mathcal{S}^k \cap H_\alpha^{-k}| = kh + 2 & \text{otherwise.} \end{cases}$$

This proves (1).

As for (2), the parallel proof works if we use the difference set

$$D_- := (\mathcal{S}^k \cap H_\alpha^k) \setminus (\mathcal{S}_{-\Sigma}^k \cap H_\alpha^k)$$

instead of D_+ . □

Proof of Theorem 1.6.

Recall that the Shi arrangements \mathcal{S}^k are free with $\exp(\mathcal{S}^k) = (1, kh, \dots, kh)$ and $\chi(\mathcal{S}^k, t) = (t-1)(t-kh)^\ell$ by [11].

Claim. Assume $\ell = 2$. Then $\mathcal{S}_{\pm\Sigma}^k$ is free if and only if either $\Sigma = \emptyset$ or $\Sigma \cap \Delta \neq \emptyset$. Therefore Theorem 1.6 holds true when $\ell = 2$.

Let us verify *Claim*. Recall $\chi_0(\mathcal{A}, t) := \chi(\mathcal{A}, t)/(t-1)$ from §2. We will apply Theorem 2.1. Define $\zeta(\Sigma) := \chi_0(\mathcal{S}_{+\Sigma}^k, 0)$ for an arbitrary subset Σ of Φ_+ . Note that $\zeta(\emptyset) = (kh)^2$. For $\alpha \in \Phi^+ \setminus \Sigma$ one has

$$(3.2) \quad \zeta(\Sigma \cup \{\alpha\}) = \zeta(\Sigma) - \chi_0(\mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}, 0)$$

because of the deletion-restriction formula for χ (i. e., [8, Corollary 2.57]). Since $\mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}$ is an arrangement in the real 2-dimensional space H_α^{-k} , we obtain

$$\chi_0(\mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}, 0) = 1 - |\mathcal{S}_{+\Sigma}^k \cap H_\alpha^{-k}|.$$

Thanks to (3.1) and (3.2), for $\alpha \in \Delta$, we have

$$\zeta(\{\alpha\}) = \begin{cases} (kh)^2 + kh & \text{if } \alpha \in \Delta, \\ (kh)^2 + (kh + 1) & \text{otherwise.} \end{cases}$$

Similarly we may verify

$$(3.3) \quad \zeta(\Sigma) = \begin{cases} (kh)^2 + kh + (kh + 1)(|\Sigma| - 1) & \text{if } \Sigma \cap \Delta \neq \emptyset, \\ (kh)^2 + (kh + 1)|\Sigma| & \text{otherwise} \end{cases}$$

by applying Proposition 3.2 and (3.2) repeatedly.

Now we will apply Theorem 2.3. Let $\mathcal{A} = \mathcal{A}(\Phi^+)$. Suppose that $m := \mathbf{1}_\Sigma$ is the indicator function of $\mathcal{A}(\Sigma)$ in \mathcal{A} . Note that the Ziegler restriction of $\mathcal{S}_{+\Sigma}^k$ onto H_z is $(\mathcal{A}, 2k + m)$. Let $(d_1, d_2) := \exp(\mathcal{A}, 2k + m)$. Define $\zeta'(\Sigma) := d_1 d_2$. Note that $\zeta'(\emptyset) = (kh)^2$. Suppose $\Sigma \neq \emptyset$. Since $\exp(\mathcal{A}, m) = (1, |\Sigma| - 1)$, Theorem 2.3 gives

$$\exp(\mathcal{A}, 2k + m) = \begin{cases} (kh + 1, kh + |\Sigma| - 1) & \text{if } \Sigma \neq \emptyset, \\ (kh, kh) & \text{if } \Sigma = \emptyset. \end{cases}$$

Thus we obtain

$$(3.4) \quad \zeta'(\Sigma) = \begin{cases} (kh + 1)(kh + |\Sigma| - 1) & \text{if } \Sigma \neq \emptyset, \\ (kh)^2 & \text{if } \Sigma = \emptyset. \end{cases}$$

Comparing the equations (3.3) and (3.4), we may conclude that $\zeta(\Sigma) = \zeta'(\Sigma)$ if and only if either $\Sigma = \emptyset$ or $\Sigma \cap \Delta \neq \emptyset$. This shows *Claim* for $\mathcal{S}_{+\Sigma}^k$ by Theorem 2.1. It is not hard to see that the parallel proof works for $\mathcal{S}_{-\Sigma}^k$.

Next assume that $\ell \geq 3$. We will apply Theorem 2.2. We still use the notation $\mathcal{A} = \mathcal{A}(\Phi^+)$ and $m = \mathbf{1}_\Sigma$. Then the Ziegler restriction of $\mathcal{S}_{\pm\Sigma}^k$ onto H_z is equal to $(\mathcal{A}, 2k \pm m)$. Theorem 2.3 shows that $(\mathcal{A}, 2k + m)$ is free if and only if $(\mathcal{A}, 2k - m)$ is free. Let $X \in L_3(\mathcal{S}_{-\Sigma}^k)$ with $X \subset H_z$. It is known that $X = Y \cap H_z$ for some $Y \in L_2(\mathcal{A})$ (see [2] for example). Note that $\Psi := \Phi \cap Y^\perp$ is a (not necessarily irreducible) root system of rank two. Then $\Psi^+ := \Phi^+ \cap Y^\perp$ is a positive system of Ψ . Define \mathcal{B}_\pm to be the restriction of $(\mathcal{S}_{\pm\Sigma}^k)_X$ to the 2-dimensional vector space Y^\perp . Then \mathcal{B}_\pm is equal to $\mathcal{S}_{\pm(\Sigma \cap \Psi^+)}^k$ when the entire root system is equal to Ψ . *Claim* shows that \mathcal{B}_+ is free if and only if \mathcal{B}_- is free. (We may easily check both \mathcal{B}_+ and \mathcal{B}_- are free for $\Psi = A_1 \times A_1$.) Note that $(\mathcal{S}_{\pm\Sigma}^k)_X = \mathcal{B}_\pm \times \{X\}$, where $\{X\}$ is a singleton arrangement in Y . Therefore \mathcal{B}_\pm is free if and only if $(\mathcal{S}_{\pm\Sigma}^k)_X$ is free. Hence $(\mathcal{S}_{+\Sigma}^k)_X$ is free if and only if $(\mathcal{S}_{-\Sigma}^k)_X$ is free. Now we may apply

Theorem 2.2 to conclude that the freeness of $\mathcal{S}_{+\Sigma}^k$ is equivalent to the freeness of $\mathcal{S}_{-\Sigma}^k$. If they are free, then $\mathcal{A}(\Sigma) = (\mathcal{A}, m)$ is also free by Theorem 2.3. Let $\exp(\mathcal{A}(\Sigma)) := (m_1, \dots, m_\ell)$. Then $\exp(\mathcal{A}, 2k \pm m) = (kh \pm m_1, \dots, kh \pm m_\ell)$. Since $(\mathcal{A}, 2k \pm m)$ is the Ziegler restriction of $\mathcal{S}_{\pm\Sigma}^k$ onto H_z , we conclude that $\exp(\mathcal{S}_{\pm\Sigma}^k) = (1, kh \pm m_1, \dots, kh \pm m_\ell)$, which completes the proof. \square

4 Proofs of Theorems 1.2 and 1.3

In this section let us prove Theorem 1.2. For that purpose, we first introduce the following lemma:

Lemma 4.1

Let $I \subset \Phi^+$ be an ideal and $X \in L_3(\mathcal{S}_{+I}^k)$ such that $X \subset H_z$. Let $\mathcal{A} = \mathcal{A}(\Phi^+)$. Choose $Y \in L_2(\mathcal{A})$ such that $X = Y \cap H_z$. Let $\Psi := \Phi \cap Y^\perp$ and $\Psi^+ := \Phi^+ \cap Y^\perp$. Then $J := I \cap \Psi^+$ is also an ideal of Ψ^+ .

Proof. Let $\alpha \in J$ and $\beta \in \Psi^+$ such that $\alpha - \beta \in \mathbb{Z}_{\geq 0}\gamma_1 + \mathbb{Z}_{\geq 0}\gamma_2$, where $\{\gamma_1, \gamma_2\}$ is the simple system of Ψ^+ . Since γ_1 and γ_2 are positive roots in Φ , $\alpha \geq \beta$ in Φ^+ . Hence $\beta \in I$, which implies that $\beta \in J$. \square

Proof of Theorem 1.2. By Theorem 1.6, it suffices to show that \mathcal{S}_{+I}^k is free. Assume $\ell = 2$. Note that $I \cap \Delta \neq \emptyset$ unless $I = \emptyset$. Hence *Claim* in the proof of Theorem 1.6 completes the proof.

Assume that $\ell \geq 3$. We apply Theorem 2.2 to prove Theorem 1.2. For that purpose, let us verify the two conditions in Theorem 2.2.

We use the notation $\mathcal{A} = \mathcal{A}(\Phi^+)$ and $m = \mathbf{1}_I$ as in the proof of Theorem 1.6. Then the Ziegler restriction of \mathcal{S}_{+I}^k onto H_z is equal to $(\mathcal{A}, 2k + m)$. By Theorem 2.4 (\mathcal{A}, m) is free, and $D(\mathcal{A}, m) \simeq D(\mathcal{A}, 2k + m)$ by Theorem 2.3. This verifies the condition (1) in Theorem 2.2.

Next we verify the condition (2). Let $X \in L_3(\mathcal{S}_{+I}^k)$ such that $X \subset H_z$. Choose Y , Ψ and Ψ^+ as in Lemma 4.1. Define \mathcal{B} to be the restriction of $(\mathcal{S}_{+I}^k)_X$ to the 2-dimensional vector space Y^\perp . Then \mathcal{B} is equal to $\mathcal{S}_{+(I \cap \Psi^+)}^k$ when the entire root system is equal to Ψ . Lemma 4.1 shows that $I \cap \Psi^+ \subset \Psi^+$ is an ideal. This verifies the freeness of \mathcal{B} because Theorem 1.2 has been already proved when $\ell = 2$. Recall that $(\mathcal{S}_{+I}^k)_X$ is free if and only if \mathcal{B} is free as we saw in the proof of Theorem 1.6. This verifies the condition (2) in Theorem 2.2. \square

Proof of Theorem 1.3.

(1) Let $k \in \mathbb{Z}_{>0}$. Define

$$A := \{\alpha - jz \mid \alpha \in \Phi^+, -k + 1 \leq j \leq k\}, \quad B := \{\alpha + kz \mid \alpha \in I\}.$$

Recall $\tilde{\text{ht}} : A \cup B \cup \{z\} \rightarrow \mathbb{Z}_{>0}$ from Theorem 1.3.

Claim 1. $\tilde{\text{ht}}(A) \subset [1, kh]$ and $\tilde{\text{ht}}(B) \subset [kh + 1, (k + 1)h - 1]$.

Let $\alpha \in \Phi^+$. For $0 < j \leq k$, we have $1 < -\text{ht}(\alpha) + jh + 1 \leq kh$ because $1 \leq \text{ht}(\alpha) < h$. For $1 - k \leq j \leq 0$, we have $1 \leq \text{ht}(\alpha) - jh < kh$ because $1 \leq \text{ht}(\alpha) < h$. This verifies $\tilde{\text{ht}}(A) \subset [1, kh]$. Let $\beta \in I$. Similarly we may easily verify $\tilde{\text{ht}}(B) \subset [kh + 1, (k + 1)h - 1]$.

Consider the standard height function $\text{ht} : \Phi^+ \rightarrow \mathbb{Z}_{>0}$. Define $g_i := |\text{ht}^{-1}(i)|$. Then $g_h = 0$ and $g_1 = \ell$.

Claim 2. $g_i + g_{h-i+1} = \ell$ for $1 \leq i \leq h$.

Recall from [10] [6] [7] that the dual partition of the pair (Φ, ht) is equal to $\exp(\mathcal{A}(\Phi^+))$:

$$\exp(\mathcal{A}(\Phi^+)) = ((1)^{g_1-g_2}, (2)^{g_2-g_3}, \dots, (h-1)^{g_{h-1}}).$$

By the duality of the exponents, we have $g_i - g_{i+1} = g_{h-i} - g_{h-i+1}$ and thus $g_i + g_{h-i+1} = g_{i+1} + g_{h-i}$ for $1 \leq i < h$. This implies that the value of $g_i + g_{h-i+1}$ does not depend upon i with $1 \leq i \leq h$. It is equal to $g_1 + g_h = \ell$. This verifies *Claim 2*.

Note that $\tilde{\text{ht}}(z) = 1$. For $i \in \mathbb{Z}_{>0}$ define $f_i := |\tilde{\text{ht}}^{-1}(i)|$. By *Claim 1*, we have $f_i = 0$ if $(k + 1)h \leq i$.

Let $1 \leq i \leq kh$. We may uniquely express $i = qh + r$ with $0 \leq q \leq k - 1$ and $1 \leq r \leq h$. Suppose that $\alpha \in \Phi^+$ and $1 - k \leq j \leq 0$. Then it is not hard to see that

$$\tilde{\text{ht}}(\alpha - jz) = i \iff j = -q \text{ and } \text{ht}(\alpha) = r.$$

Next we assume that $0 < j \leq k$. Then it is not hard to see that

$$\tilde{\text{ht}}(\alpha - jz) = i \iff j = q + 1 \text{ and } \text{ht}(\alpha) = h - r + 1.$$

Now we may conclude

$$f_i = \begin{cases} 1 + g_1 + g_h = \ell + 1 & \text{if } i = 1, \\ g_r + g_{h-r+1} = \ell & \text{if } 1 < i \leq kh. \end{cases}$$

thanks to *Claim 2*. Thus we obtain

$$f_1 - f_2 = 1, \quad f_2 - f_3 = f_3 - f_4 = \dots = f_{kh-1} - f_{kh} = 0.$$

Next let $kh < i < (k + 1)h$. We may uniquely express $i = kh + r$ with $1 \leq r \leq h$. Suppose that $\beta \in I$. Then it is not hard to see that

$$\tilde{\text{ht}}(\beta + kz) = i \iff \text{ht}(\beta) = r.$$

Define $p_i = |\{\beta \in I \mid \text{ht}(\beta) = i\}|$ for $1 \leq i \leq h$. Then we conclude that $f_i = p_r$ when $kh < i = kh + r < (k+1)h$. Thus we obtain

$$\begin{aligned} f_{kh} - f_{kh+1} &= \ell - p_1, f_{kh+2} - f_{kh+3} = p_2 - p_3, \\ f_{kh+3} - f_{kh+4} &= p_3 - p_4, \dots, f_{kh+h-1} - f_{kh+h} = p_{h-1} - p_h. \end{aligned}$$

Recall Theorem 2.4 which asserts that

$$\exp(\mathcal{A}(I)) = (m_1(I), \dots, m_\ell(I)) = ((0)^{\ell-p_1}, (1)^{p_1-p_2}, (2)^{p_2-p_3}, \dots).$$

Therefore the dual partition of the pair $(A \cup B \cup \{z\}, \widetilde{\text{ht}})$ is equal to

$$((0)^{\ell+1-f_1}, (1)^{f_1-f_2}, (2)^{f_2-f_3}, \dots, \dots) = (1, kh + m_1(I), \dots, kh + m_\ell(I)).$$

This proves Theorem 1.3 (1) because of Theorem 1.6.

(2) The parallel proof works for \mathcal{S}_{-I}^k . □

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